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# Corfu 05 lectures - part I: Strings on curved backgrounds

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## Abstract.

In these introductory lectures we summarize some basic facts and techniques about perturbative string theory (sections 1 to 6). These are further developed (sections 7 and 8) for describing string propagation in the presence of gravitational or gauge fields. We also remind some solutions of the string equations of motion, which correspond to remarkable (NS or D) brane configurations.

A part II by Emilian Dudas will be devoted to orientifold constructions and applications to string model building.

## 1. Fields versus strings

Field theories contain in general an arbitrary number of particles (fields) and their mutual interactions are essentially constrained only by the requirements of renormalizability and unitarity. This results in a large freedom in choosing these interactions (Yang–Mills, Yukawa,  $\phi^4$  just to name some of the best studied) that is in practice only limited by phenomenological constraints. Enough freedom remains to incorporate some luxury items such as grand-unified groups, supersymmetry or Kaluza–Klein spectra at least if they do not contradict the available experimental data. This is nevertheless not enough to introduce gravity in the picture, at least unless we choose to abandon some commonly accepted rules.

A possible way out (and possibly the only really promising one at this time) is provided by string theory. In this case an infinite spectrum of particles arises but this time naturally arranged in representations of a superalgebra and defined in dimension  $D > 4$ . This implies the presence of a Kaluza–Klein spectrum. Among the highlights there is the fact that – and this is different from the field case – gauge interactions are not added *ad hoc* but do appear naturally, and together with them gravitational interactions, in the form of supergravity. Moreover the presence of large gauge groups is in some way natural.

Of course many open questions still remain. Among them the fact that the little freedom allowed by the consistency constraints in ten dimensions becomes pretty large in four. Other still

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open problems concern the breaking of supersymmetry or the role that string theory can have in explaining the cosmological evolution. An important feature (or, a weakness, depending on the point of view) is the fact that string theory incorporates more exotic objects. In particular the theory is only consistent if we add extended objects, like  $Dp$  branes or NS5 branes and higher-order forms. The latter are generalization of the field-theory gauge fields and appear in the Ramond–Ramond sector of the type IIA and IIB theories where they are coupled to  $Dp$  branes<sup>2</sup> and in the universal sector in terms of a three-form  $H_{[3]}$  that is electrically coupled to the string and magnetically to the NS5 brane. Although originally designed around flat space, these branes can propagate in curved (gravitational) backgrounds with various gauge fields switched on (fluxes through the compact sector of the geometry) and have become an almost indispensable tool for probing aspects such as string gravity, black hole (thermo)dynamics, supersymmetry breaking, moduli stabilization, in analyzing holography and decompactification regimes and – last but not least – in the quest for time-dependent solutions with a cosmological interest.

A fact – at the root of many of the present complications – is that strings in general prefer spheres  $S^n$  and anti-de Sitter spaces  $\text{AdS}_n$  (it suffices to recall the partial breaking of supersymmetry or holography, that naturally bridges strings and super-Yang–Mills theories) while do not seem to like hyperbolic planes  $H_n$  or de Sitter spaces  $\text{dS}_n$  – which is a sort of archetype of the difficulties that one encounters while trying to make contact with cosmology.

The purpose of the present lecture notes is to summarize some of the basic tools, necessary for addressing the latter issues. Sections 2 and 4 deal with the basics on perturbative closed strings which are extended to open strings in Sec. 4.3. Once the massless string spectra are established, coherent states of massless excitations can be generated, which act as classical backgrounds for the corresponding fields (section 5). String propagation in such environments is further analyzed in section 6.

These notes are elementary and many important issues are missing. For this reason we avoid referring to the original papers in the course of the main text. A list of references for further reading is made available at the end<sup>3</sup>.

## 2. String motion in $D$ -dimensional flat spacetime

### 2.1. Free-falling relativistic particle

Let us start from the most simple system: the motion of a free-falling relativistic point particle with mass  $m \neq 0$ . The manifest relativistic action is just given by the length of the trajectory between the two extremal fixed points in spacetime:

$$S = -m \int_i^f d\tau \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} = -m \int_i^f ds. \quad (1)$$

Since the Compton wavelength is defined as  $\lambda = \hbar/mv$  we expect a classical behaviour for large  $m$  and quantum behaviour for small  $m$ , where “large” and “small” are defined with respect to some energy scale of the system.

As it stands this system is overdetermined. In fact, in a Hamiltonian formalism (not manifestly relativistic) the motion  $\mathbf{x} = \mathbf{x}(\tau)$  can be described in a  $2(D-1)$ -dimensional phase space defined by:

$$\begin{cases} \tau = x^0 \\ p^\mu p_\mu = -m^2. \end{cases} \quad (2)$$

<sup>2</sup> In the same way as the electromagnetic field, which is a two-form, is electrically coupled to a point particle, a  $(p+2)$ -form is electrically coupled to a  $Dp$  brane

<sup>3</sup> Further reviews on string theory can be found at the following URL <http://www.slac.stanford.edu/spires/find/hep/www?hn=pre+reviewIf>

This is the reason why, apart from being Poincaré invariant, the action is also reparameterization-invariant. This local symmetry can be used to remove one degree of freedom by imposing  $\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$  (that is equivalent to  $p^\mu p_\mu = -m^2$ ) and still we will be left with the residual time translation invariance  $\tau \rightarrow \tau + a$  which allows to remove the initial value for the time coordinate.

The presence of redundant (spurious) degrees of freedom is a common trait of manifest relativistic invariant theories. The standard example is given by the electromagnetic potential four-form  $A_\mu$  which contains two unphysical polarizations as a consequence of the gauge invariance  $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ . They can be removed, *e.g.* by going to the light-cone gauge, but the price to pay is the loss of manifest Lorentz invariance.

## 2.2. Strings

Let us now move to strings. The natural extension of the previous description is provided by the Nambu–Goto action measuring the area of the world-sheet swept by the string in its motion from an initial configuration  $i$  to a final configuration  $f$ :

$$S = -T \int_i^f d^2\zeta \sqrt{-\det(\hat{g})} = -T \int_i^f dA, \quad (3)$$

where  $\hat{g}$  is the pull-back of the spacetime metric on the world-sheet (in components  $\hat{g}_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ ), parameterized by the variables  $\zeta^\alpha, \alpha = 1, 2$ . The parameter  $T$  is the string tension, whose length is not fixed but depends on the configuration. In average this is given by  $\ell \sim 1/\sqrt{T} = \sqrt{2\pi\alpha'}$ . This new parameter  $\alpha'$  fixes the scale and hence the behaviour of the string: classical when  $\alpha'$  is small and quantum when  $\alpha'$  is large with respect to the typical energies of the system. Just to have an order of magnitude,  $\alpha'$  is of the order of  $1/(10^{19}\text{GeV})^2$  (the inverse square of the Planck mass). For this reason it can be used as the “small parameter” in a perturbative expansion. We will distinguish between *exact solutions*, known to all orders in  $\alpha'$ , including non-perturbative effects and *approximate solutions* for which, in general, only the first terms of the development are known. The other possible expansion parameter  $g_{\text{st}}$ , controlling the string topological expansion, will be introduced in the next chapter.

The action (3) is Poincaré invariant and it is not difficult to show its reparameterization invariance symmetry  $\zeta^\alpha \rightarrow \tilde{\zeta}^\alpha(\zeta^\beta)$ . The latter plays a fundamental role in string theory at many levels, including some aspects that we do not yet fully understand (*e.g.* the full spacetime gauge invariance which is a guide for understanding the issues of second quantization). Just as before this local symmetry (or, equivalently, the system being overconstrained) introduces a redundancy in the degrees of freedom where two coordinates can be used as world-sheet parameters. Of course a light-cone-like gauge can be used in which the physical degrees of freedom are the only ones appearing again at the expense of the manifest Lorentz invariance.

The classical action is just the first step: although canonical quantization of the string can be performed successfully in this formalism, the square root poses some problems in the path-integral quantization and in the generalization to superspace. This is why Brink, Di Vecchia, Howe, Deser, Zumino introduced the Polyakov action (named after the latter for having analyzed its path-integral quantization). The approach is reminiscent of Lagrange multipliers, *i.e.* consists in introducing a new auxiliary field and a new local symmetry in order to have a classically equivalent dynamics. This action reads:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\zeta \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{\lambda}{4\pi} \int d^2\zeta \sqrt{-h} R \quad (4)$$

where  $h$  is the extrinsic metric of the world-sheet and  $R$  the scalar curvature. The reparameterization invariance is supplemented with the Weyl invariance  $h_{\alpha\beta} \rightarrow e^{2\rho} h_{\alpha\beta}$ . The Einstein–Hilbert action in two dimensions is purely topological: this is why  $h$  is non-dynamical

and can hence be locally set to  $h_{\alpha\beta} = \delta_{\alpha\beta}$ .

In terms of the equations of motion, the Weyl invariance is equivalent to the constraint

$$\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2} \eta_{\alpha\beta} (\partial_\gamma X^\mu \partial^\gamma X^\nu) \eta_{\mu\nu} = T_{\alpha\beta} = 0 \quad (5)$$

that must be satisfied by a physically sensible solution to the equations of motion. In the usual notation one introduces the variables  $(\sigma, \tau) = \zeta^\alpha$  and the left- and right-moving light-cone parameters  $\zeta^\pm = \tau \pm \sigma$ . In these terms the equations of motion read  $\partial_+ \partial_- X^\mu = 0$  and the general solution can be easily cast in a Fourier-expansion form as the sum of the center of mass motion and a series of harmonics:

$$X^\mu(\sigma, \tau) = x_0^\mu + 2\alpha' p^\mu \tau + \begin{cases} \sum_{n \neq 0} i \sqrt{\alpha'/2} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau-\sigma)} (\text{right sector}) \\ \sum_{n \neq 0} i \sqrt{\alpha'/2} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} (\text{left sector}) \end{cases} \quad (6)$$

with the conditions  $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$  and  $\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$ . The constraints  $T_{\alpha\beta}$  translate into constraints over  $p^\mu, \alpha_n^\mu, \tilde{\alpha}_n^\mu$  that are therefore not independent. One possible choice consists in eliminating two light-cone directions  $X^\pm$  which are then expressed as functions of the others. As a result  $x_0^\pm, x_0^i, p^\pm, p^i$  and  $\alpha_n^i, \tilde{\alpha}_n^i$  remain independent modulo the two (mass-shell and level-matching) constraints:

$$M_{\text{string}}^2 \equiv -p_\mu p^\mu = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^j \delta_{ij} = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^j \delta_{ij}. \quad (7)$$

The latter expression defines the mass squared of the string in a given configuration. The classical mass spectrum of the bosonic string is a continuum above a zero-mass ground state.

### 3. Basics on string quantization in $D$ -dimensional flat spacetime

Different options exist for quantizing constrained systems with gauge symmetries. These include:

- light-cone quantization, which consists in considering only the physical degrees of freedom. This automatically guarantees unitarity but Lorentz invariance is not manifest;
- covariant quantization in which all the degrees of freedom are taken into account. Consistency conditions must be imposed, and hopefully remove negative-norm states from the spectrum.

Other possible choices include path integral quantization, BRST quantization, etc. In this chapter we will concentrate on the light-cone canonical and the path-integral quantizations.

#### 3.1. Canonical quantization

Let us consider the light-cone (canonical) quantization starting from the Fourier expansion in Eq. Equation (6). The zero-modes satisfy the usual commutation relations:

$$[x^\mu, p^\nu] = i \eta^{\mu\nu}, \quad (8)$$

and  $\alpha_{n>0}^{\dagger\mu}, \tilde{\alpha}_{n>0}^{\dagger\mu}$  have the roles of creation operators. A general state is written as:

$$|\mathbf{p}, i_r, m_r, \dots, j_s, n_s, \dots\rangle = (\alpha_{m_r}^{i_r} \dots) (\tilde{\alpha}_{n_s}^{j_s} \dots) |\mathbf{p}\rangle, \quad (9)$$

where  $i_r, j_s = 1, 2, \dots, D-1$ . The left and right levels are

$$N = \sum_r m_r, \quad \bar{N} = \sum_s n_s, \quad (10)$$

so that the level-matching and mass-shell conditions read:

$$N = \bar{N}, \quad (11)$$

$$-p^2 = M^2 = \frac{4}{\alpha'} \left( N - \frac{D-2}{24} \right), \quad (12)$$

where the  $(D-2)/24$  is the quantum two-dimensional vacuum energy. It is useful to point out that each state is a one-particle state in some representation of the Poincaré group. Hence, the Hilbert space spans an infinity of different Poincaré representations, massless and massive with a mass of the order of the Planck mass ( $10^{19}$  GeV). First-quantized strings are described in terms of a second-quantized field theory. This is at the moment the state of the art and no satisfactory extension is yet known. In some sense this is the same situation in which the interaction between protons and electrons was before the introduction of QED: propagators, vertices and a collection of ad-hoc perturbation rules. Actually, string theory is in this respect more satisfactory since it is very constrained (practically all aspects are frozen, even the number of dimensions), including the way in which string states (the equivalent of particles) interact.

Let us now give a closer look at the spectrum:

- the ground level  $|\mathbf{p}\rangle$  has mass:

$$-p^2 = M^2 = -\frac{D-2}{24}, \quad (13)$$

which is tachyonic (and hence unstable<sup>4</sup>) unless  $D = 2$ ;

- the first level is:

$$|\mathbf{p}, 1, i, 1, j\rangle = \alpha_{-1}^i \tilde{\alpha}_{-1}^j |\mathbf{p}\rangle \quad (14)$$

and has mass

$$-p^2 = M^2 = 1 - \frac{D-2}{24}. \quad (15)$$

This is a symmetric tensor with  $(D-2) \times (D-2)$  degrees of freedom which we can decompose in its traceless symmetric part ( $D(D-3)/2$  dof), antisymmetric ( $(D-2)(D-3)/2$ ) and trace (1 dof)<sup>5</sup>. Since only the transverse degrees of freedom appear here, we obtain a representation of the Poincaré group if the particle is massless, that is if  $D = 26$ . This tensor is part of the universal sector of string theory, *i.e.* it appears in the massless sector of every model;

- higher levels are massive, bosonic representations.

Let us pause for a moment and discuss this latter result. Bosonic string theory naturally contains a critical dimension  $D = 26$ . This can be interpreted in various ways. In the light-cone quantization one can show that the conserved charges associated to the Lorentz currents do not close if  $D \neq D_{\text{cr}}$ , *i.e.* there is an anomaly: the Lorentz algebra is not only non-manifest but not present altogether. In a covariant quantization scheme the critical dimension is required for unitarity. The path integral quantization allows for a different point of view on the critical dimension and brings extra information about the perturbative expansion and the dynamics.

<sup>4</sup> The presence of tachyons usually means that the theory is studied around a false vacuum. Theories with tachyons have usually been discarded because of the lack of a non-perturbative description, but the situation is lately changing.

<sup>5</sup> These three components will be later identified with the graviton, the Kalb–Ramond field and the dilaton. Such an identification requires the treatment of interactions.

### 3.2. Path integral quantization: the string perturbative expansion

In the path integral quantization, the partition function – or any correlator – is written as an integral over the embedding coordinates  $X^\mu$  and the non-dynamical two-dimensional metric  $h_{\alpha\beta}$ :

$$Z = \int \frac{\mathcal{D}X^\mu \mathcal{D}h_{\alpha\beta}}{[\text{Volume of diffeomorphism and Weyl groups}]} e^{-S[X^\mu, h_{\alpha\beta}]}. \quad (16)$$

The minus sign in the exponential is consequence of the Polyakov prescription for a Wick rotation on the world-sheet. Out of this we can derive important results about the string loop expansion and the critical dimension.

### 3.3. String loop expansion

The integral over  $h_{\alpha\beta}$  can be decomposed as a sum over the topologies of the two-dimensional world-sheet and an integral over the metrics with fixed topology:

$$\int \mathcal{D}h_{\alpha\beta} = \sum_{\text{topologies}} e^{-\lambda\chi} \int_{\text{fixed topology}} \mathcal{D}h_{\alpha\beta}, \quad (17)$$

where  $\chi$  is the Euler number

$$\chi = \frac{1}{4\pi} \int d^2\zeta \sqrt{-h} R = 2 - 2\gamma - M, \quad (18)$$

$\gamma$  being the genus and  $M$  the number of boundaries. An important consequence of this analysis is the appearance of a *unique* string vertex with coupling constant

$$e^\lambda = g_{\text{closed string}}. \quad (19)$$

The various features that have emerged sofar exhibit major differences with ordinary field theory:

- the spectrum of particle, *i.e.* the string states belonging to a given Poincaré representation, is given once and forever;
- the interactions are fixed by the Polyakov path integral quantization;
- the string coupling is (at least in principle) dynamically fixed:  $\lambda$  turns out to be the vacuum expectation value of the dilaton;
- each string state corresponds to a *vertex operator*, *i.e.* an operator of the two-dimensional theory that we insert to compute the amplitudes;
- even if a non-perturbative formulation is not available, we still have a handle on non-perturbative effects with respect to  $g_{\text{st}}$ . With this we mean that in some situation we can use dualities to map a problem in the  $g_{\text{st}} \gg 1$  regime to another problem with coupling  $1/g_{\text{st}} \ll 1$  which can then be tackled with perturbative techniques.

A last remark is in order: the very concept of world-sheet is by definition semiclassical and breaks down in some regimes, controlled by  $g_{\text{st}}$ . At this moment we still miss an appropriate non-perturbative description from which the Polyakov prescription would follow.

### 3.4. Critical dimension: critical versus non-critical strings

In the approach of canonical light-cone quantization the critical dimension  $D = 26$  appears to guarantee that transverse-two-tensor modes are massless. A different interpretation for this phenomenon, is available from the path-integral viewpoint. The measure appearing in the integral of Eq. (16) is not Weyl invariant: there is a quantum anomaly which is an obstruction to the decoupling of the two-dimensional metric, and is reflected in the central charge of the two-dimensional CFT. Two different consistent string regimes finally emerge:

- for  $D = 26$  (critical strings),  $h_{\alpha\beta}$  is not dynamical;
- for  $D < 26$ , the scale factor of  $h_{\alpha\beta}$  contributes to the spectrum as the Liouville mode.

Liouville theory was studied in the early eighties as a two-dimensional field theory. It reappeared in the nineties in the developments of *matrix models* and non-perturbative two-dimensional quantum gravity and has again attracted some attention recently in the framework of holography.

## 4. Advanced string motion: extra degrees of freedom and open strings

### 4.1. Fermionic degrees of freedom

The states we built in Sec. 2 are by construction only integer-spin representations of the Poincaré algebra, which implies that in order to incorporate fermions we need to add extra degrees of freedom. This can be done following two types of formalisms:

- the *Green–Schwarz* where spacetime spinors are introduced with manifest spacetime supersymmetry. This formalism is heavy; it turns out to be convenient only in some specific situations.
- the *Neveu–Schwarz–Ramond* which can be used more generally, has no explicit spacetime supersymmetry and does not allow a general treatment of the Ramond–Ramond fields in terms of sigma-model.

Let us introduce the *fermionic coordinates*  $\psi^\mu$  and a sort of superspace described by couples  $(x^\mu, \psi^\mu)$ . The string motion is captured by a set of functions:

$$\begin{cases} x^\mu = X^\mu(\zeta), \\ \psi^\mu = \Psi^\mu(\zeta). \end{cases} \quad (20)$$

The two-dimensional Majorana spinors appear as world-sheet fields and we need to specify their dynamics<sup>6</sup>. The natural choice consists in adding to the action (Eq. Equation (4)) an ordinary Dirac massless term:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\zeta \sqrt{-h} \left( h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - i \bar{\psi}^\mu \rho^a e^\alpha_a D_\alpha \psi^\nu \right) \eta_{\mu\nu}, \quad (21)$$

where  $h$  is the two-dimensional metric,  $\rho^a$  the two-dimensional Dirac  $\gamma$  matrices and  $e^\alpha_a$  the *zwei-bein*. This action has the following symmetries: local Weyl-rescaling of the two-dimensional metric, two-dimensional diffeomorphism invariance, global two-dimensional  $N = (1, 1)$  supersymmetry and global spacetime Poincaré invariance.

In spite of its high symmetry, the action at hand is incomplete. This caveat emerges during the quantization. In fact, using the canonical mode expansion for  $\psi^\mu$  and imposing the anticommutation relations one witnesses the appearance of negative-norm states which would decouple if we could go to a light-cone gauge and eliminate say *e.g.*  $\Psi^0$  and  $\Psi^{D-1}$  just as we did for  $X^0$  and  $X^{D-1}$ . This is possible in the bosonic sector thanks to the Weyl and *diff* local

<sup>6</sup> It is worth to point out that introducing world-sheet fermions does not automatically guarantee the existence of spacetime spinors.

invariance which for the action above do not have any counterpart in the fermionic sector. The way out then consists in promoting the global  $N = (1, 1)$  supersymmetry to a superconformal  $N = (1, 1)$  supergravity. We must therefore associate to the two-dimensional graviton  $h_{\alpha\beta}$  a two-dimensional gravitino superpartner  $\chi^\alpha$ . Both fields are non-dynamical: the graviton because of the Weyl and diffeomorphism invariance, the gravitino because of super-Weyl and local supersymmetry. Both will then contribute to the anomaly in opposite directions, thus changing the critical dimension:

$$D + \frac{D}{2} - 26 + 11 = 0 \Rightarrow D = 10. \quad (22)$$

#### 4.2. GSO projection and spectrum

We can now consider the overall spectrum. Let us call  $\alpha_n^\mu$  and  $\tilde{\alpha}_n^\mu$  the left- and right-moving bosonic oscillators and  $\beta_m^\mu$  and  $\tilde{\beta}_m^\mu$  the fermionic ones. Being fermions, we have a choice on the boundary conditions that can be either periodic or anti-periodic:

- in the Neveu–Schwarz sector the conditions are anti-periodic and the  $m$ 's are half-integer;
- in the Ramond sector the conditions are periodic and the  $m$ 's are integer. In particular this allows for the presence of a zero-mode for which

$$\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}. \quad (23)$$

Depending on the sector, the ground state is respectively a spacetime scalar or a Majorana–Weyl spinor.

Although the string spectrum is in general very constrained there are ways to consistently remove or add (replace) full sectors containing an infinite number of states. This is in general possible in presence of discrete symmetries such as world-sheet parity  $\sigma \rightarrow -\sigma$ , left fermionic number  $(-)^{F_L}$  or right fermionic number  $(-)^{F_R}$  (these are all  $\mathbb{Z}_2$  symmetries). Any projection must be shown to be consistent, *i.e.* the remaining states must form a closed set and if new states are created they must be added in the form of new sectors. In particular, the GSO (Gliozzi–Scherk–Olive) projection allows to consistently remove a large number of sectors, including the tachyonic ones. Possible consistent projections are:

- type 0A and type 0B theories which have tachyons and no spacetime fermions
- type IIA and type IIB theories without tachyons, with spacetime bosons and fermions, and 32 spacetime supercharges in ( $N = 2$  supersymmetry in ten dimensions). In particular in IIA the supersymmetry is non-chiral whereas it is chiral in IIB.

#### 4.3. Open strings

Let us go back for a moment to the simple situation of a bosonic string described by the Nambu–Goto action:

$$S = -T \int d^2\zeta \sqrt{\det(\hat{g})}, \quad \hat{g}_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \quad (24)$$

As it is usually the case, the variation  $\delta S$  contains boundary terms that we discarded in the previous derivation of the Euler–Lagrange equations:

$$\delta S_{\text{surface}} = -T \int_{\text{initial}}^{\text{final}} \{ \partial_\sigma X_\mu \delta X^\mu|_{\sigma=\pi} - \partial_\sigma X_\mu \delta X^\mu|_{\sigma=0} \}. \quad (25)$$

These must however vanish:

- for closed strings  $X^\mu$  is periodic and  $X^\mu|_{\sigma=\pi} = X^\mu|_{\sigma=0}$  so the term is identically zero;



- for open strings the two endpoints are independent and both the addends variations must vanish for an arbitrary variation.

A term of the form  $\partial_\sigma X_\mu \delta X^\mu$  can vanish in two ways, *i.e.* each of the two factors can be zero. In the general case  $p + 1$  directions are Neumann, *i.e.* satisfy:

$$\partial_\sigma X_\mu|_{\sigma=\text{endpoint}} = 0, \quad \mu = 0, 1, \dots, p, \quad (26)$$

and the  $9 - p$  remaining are Dirichlet, *i.e.* satisfy:

$$\delta X^\mu|_{\sigma=\text{endpoint}} = 0, \quad \mu = p + 1, p + 2, \dots, 9. \quad (27)$$

A special case is given by  $p = 9$ , *i.e.* when we have one D9-space-filling brane. This is the so-called *traditional open string*

$$\partial_\sigma X_\mu|_{\sigma=0,\pi} = 0, \quad \mu = 0, 1, \dots, 9, \quad (28)$$

and in terms of harmonic oscillators this condition implies

$$\alpha_n^\mu = \tilde{\alpha}_n^\mu. \quad (29)$$

This is as close as we can get to the closed string case; in particular the vacuum state is unchanged:  $|\mathbf{p}\rangle$  for the bosonic case and when adding fermions  $|\mathbf{p}\rangle_{NS}$  is scalar and  $|\mathbf{p}\rangle_R$  is a Majorana-Weyl spinor. Only one set of (bosonic or fermionic) oscillators act on these vacua. The resulting two-dimensional theory is  $N = 1$  locally supersymmetric, *i.e.* has an  $N = 1$  superconformal supersymmetry (the difference with respect to the previous case is that one of the supersymmetries is broken by the boundary conditions). In some sense the spectrum is the “square root” of the closed-string one.

The massless spectrum contains an NS bosonic part given by a spacetime vector with  $D-2 = 8$  physical components and a R fermionic consisting in a spacetime Majorana-Weyl spinor. They compose the vector multiplet of type I  $N = 1$  supersymmetry algebra in ten dimensions.

The present analysis calls for some remarks:

- gravity is missing from the picture but this is not surprising since open strings by themselves do not constitute a consistent theory (they cannot merge to make closed strings). In other words a consistent theory has to be a collection of closed and open sectors;
- there is an operation  $\Omega : \sigma \rightarrow \pi - \sigma$  which changes the orientation of the string. States are even or odd with respect to this  $\mathbb{Z}_2$ ;
- type IIA and IIB in ten dimensions do not have massless vectors in their perturbative spectrum. And when compactified give rise to Abelian  $U(1)$ ’s;
- if we take a bosonic left sector and a fermionic right one we obtain the heterotic theory with  $N = 1$  spacetime supersymmetry. Until the mid nineties heterotic string was the standard framework for phenomenology since it allows for non Abelian gauge groups:  $E_8 \times E_8$  or  $SO(32)$ .

The landscape of string theory has been drastically changed with the appearance of D branes. Nowadays most of the attempts to make contact with phenomenology are based on type I and type II theories in presence of D branes.

Dirichlet branes are not rigid static objects. They have their own dynamics (inherited by the attached open strings); moreover the end points of open strings carry a  $U(1)$  charge that couples to the  $U(1)$  gauge field present on their spectrum. This  $U(1)$  field penetrates the D brane and influences the dynamics. D branes have their own spectrum that demands a quantum mechanical description. This can be obtained via the boundary conformal field theory BCFT of the open string. A semiclassical approximation is nevertheless possible at low energies (the small parameter being  $1/\alpha'$  via the Dirac–Born–Infeld action). Being extended objects, D branes break the translational invariance and therefore part of the spacetime supersymmetry, which is an asset from the point of view of phenomenology.

## 5. Strings in background fields

### 5.1. What is a background field?

In QED it is customary to study the classical field created by a point-like charge and quantize the theory for test particles in presence of such a background field. This is by its very nature a semiclassical approximation because the background satisfies the Maxwell equations. At the same time under certain regimes the truly quantum nature of the system is bound to show up. Contact between the classical relations and the quantum theory is possible by describing the background fields as vacuum expectation values of the field operators in coherent states.

A similar path can be followed in string theory where, although we miss a full formalism similar to QED, as we have already stressed above, we nevertheless have in flat space a first-quantized version and a set of consistent and well-defined perturbative rules.

Starting from the massless excitations we can build coherent states and interpret them as background fields. The (semi)classical interpretation is then possible under the condition of not going too deep, *i.e.* only in a perturbative regime identified by the scales  $\alpha'$  and  $g_{\text{st}}$ . We will only look at massless fields since the massive ones would have a very short range and would take us away from the classical background field approximation from the very beginning.

To be more concrete we need to identify which elementary objects are the sources for these backgrounds and what are the equations that the semiclassical fields satisfy – or equivalently which is the low-energy effective action for the massless content of the string spectrum. This means that the analysis of string theory in non-trivial backgrounds will be important both for probing new environments compatible with string dynamics potentially relevant for phenomenology, and for going off-shell at least for the massless excitations in some chosen regimes.

### 5.2. Sources for antisymmetric fields

To answer the first question, *i.e.* what are the elementary objects acting as sources for the background fields, we must consider the massless component of the spectrum. These are:

- the universal NS sector,  $G_{\mu\nu}$ ,  $B_{\mu\nu}$  and  $\Phi$ ;
- the open string gauge field  $A_\mu$ ;
- the RR forms which are  $F_{[2]} = dA_{[1]}$  and  $F_{[4]} = dA_{[3]}$  in type IIA and  $F_{[1]} = dA_{[0]}$ ,  $F_{[3]} = dA_{[2]}$  and  $F_{[5]} = dA_{[4]}$  in type IIB.

In the case of the electromagnetic field (a two-form) the action is written as

$$S = -\frac{1}{4\kappa^2} \int d^D x F_{\mu\nu} F^{\mu\nu}. \quad (30)$$

The natural classical electric source for such a field is provided by a point-like charge. The action is then given by:

$$\begin{aligned} S_{\text{int}}^{\text{el}} &= q \int_{\text{trajectory}} A = q \int A_\mu(X(\tau)) \frac{dX^\mu}{d\tau} d\tau = \\ &= q \int d\tau \left( \int d^D x \delta^D(X(\tau) - x) A_\mu(x) \right) \frac{dX^\mu}{d\tau} = \int d^D x A_\mu(x) j^\mu(x), \end{aligned} \quad (31)$$

where we have defined the electric current associated with the point-like charge

$$j^\mu(x) = \int d\tau q \delta^D(X(\tau) - x) \frac{dX^\mu}{d\tau}. \quad (32)$$

type	form	electric source	magnetic source
IIA	$F_{[2]}$	D0	D6
IIA	$F_{[4]}$	D2	D4
IIB	$F_{[1]}$	D(-1)	D7
IIB	$F_{[3]}$	D1	D5
IIB	$F_{[5]}$	D3	D3

**Table 1.** Field strengths and sources in type II strings

The dual magnetic field is a  $(D - 2)$ -form  $*F = \tilde{F}$  generated by a  $(D - 3)$ -form potential  $\tilde{A}$ . This couples to a classical magnetic source, which is an extended object with  $(D - 3)$ -dimensional world-volume, or a  $(D - 4)$  brane.

$$S_{\text{int}}^{\text{mag}} = q_m \int_{\text{world-volume}} \tilde{A} = \int d^D x \tilde{A}_{\mu_1 \mu_2 \dots \mu_{D-3}} \tilde{j}^{\mu_1 \mu_2 \dots \mu_{D-3}}, \quad (33)$$

where  $\tilde{j}$  is the magnetic current associated with the  $(D - 4)$  brane.

For a more general  $(p + 2)$ -form  $F_{[p+2]}$  the action reads:

$$S = -\frac{1}{(p + 2)! \kappa^2} \int d^D x F_{\mu_1 \mu_2 \dots \mu_{p+2}} F^{\mu_1 \mu_2 \dots \mu_{p+2}}. \quad (34)$$

Generalizing the construction above, one sees that the natural elementary classical electric sources for such a field are objects with a  $(p + 1)$ -dimensional world-volume, *i.e.*  $p$  branes:

$$S_{\text{int}}^{\text{el}} = \int_{\text{world-volume}} A_{[p+1]} = \int d^D x A_{\mu_1 \mu_2 \dots \mu_{p+1}} j^{\mu_1 \mu_2 \dots \mu_{p+1}}, \quad (35)$$

whereas the magnetic sources are  $(D - p - 4)$  branes:

$$\int \tilde{A}_{[D-p-3]} = \int d^D x \tilde{A}_{\mu_1 \mu_2 \dots \mu_{D-p-3}} \tilde{j}^{\mu_1 \mu_2 \dots \mu_{D-p-3}}. \quad (36)$$

In string theory, there exist a plethora of massless modes in antisymmetric representations that can combine in coherent states so to give the desired background. They are

- ordinary  $U(1)$  gauge fields  $A_\mu$  from the open string;
- the antisymmetric NS tensor  $H_{[3]} = dB_{[2]}$  which is electrically coupled to the fundamental string itself and magnetically coupled to a  $(10 - 1 - 4 = 5)$  brane (the so-called NS5 brane);
- in type IIA and IIB the various RR fields coupled as in Tab. 1.

### 5.3. Generalizing the Polyakov action

Up to this point we have only dealt in detail with strings propagating in a trivial, flat background. This is not sufficient because the string carries energy and electric charge with respect to a 3-form field strength  $H_{[3]}$ . In other words it couples to the metric  $G_{\mu\nu}$ , the dilaton  $\Phi$  and the Kalb–Ramond field  $B_{\mu\nu}$  (*i.e.* the two-form potential for  $H_{[3]}$ ).

A generalization of the Polyakov action (Eq. Equation (4)) is needed in order to capture these more general situations. The most general action compatible with two-dimensional renormalizability, Weyl and diffeomorphism invariance reads:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\zeta \sqrt{-h} \left( h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) - \alpha' \Phi(X) R \right). \quad (37)$$

Although this action looks natural, it is not clear that it could be obtained from first principles, based on a coherent-state approach. A satisfying bottom-up approach should take into account the fact that, when quantized around flat space, strings exhibit quanta of  $G_{\mu\nu}$ ,  $B_{\mu\nu}$  and  $\Phi$  fields interacting in a very definite pattern. The way in which those fields get further organized into coherent states is by no means arbitrary and the way in which those, which we can read as classical or semiclassical fields, couple to the string itself is also definite.

This programme can be realized, with the following remarks though:

- as it is usually the case, there is an underlying assumption about the validity of the approximation which descends from the string topology expansion. This is crucial here since it becomes clear that the coupling  $g_{\text{st}}$  is related to the VEV of the dilaton as

$$g_{\text{st}} = e^\lambda, \quad \lambda = \langle \Phi \rangle \quad (38)$$

and hence  $\lambda$  is in general a function of the position: it is then possible to obtain a motion in the target space bridging a perturbative regime to a non-perturbative one;

- the fields  $G_{\mu\nu}$ ,  $B_{\mu\nu}$  and  $\Phi$  cannot be arbitrary since, at least in principle, they are coherent states of elementary strings states. In particular we shall demand that the same constraints of Weyl and diffeomorphism invariance are satisfied at the quantum level. In this case those requirements will not only impose a critical dimension but will translate into equations for the semiclassical fields;
- $G_{\mu\nu}$ ,  $B_{\mu\nu}$  and  $\Phi$  appear as parameters in the two-dimensional theory, in general receive quantum corrections depending on the renormalization scheme and can be known exactly or only to some approximation in the parameter  $\alpha'$ ;
- although  $G_{\mu\nu}$  is a metric, its very geometric interpretation becomes questionable wherever the local curvature  $R$  is large with respect to  $1/\alpha'$ . This is due to the extension of the string ( $\sim \sqrt{\alpha'}$ ), which probes  $G_{\mu\nu}$ .

#### 5.4. Equations of motion

The procedure for imposing Weyl invariance can be carried on in different ways: compute amplitudes and demand the decoupling of the conformal factor (Liouville mode) or compute the  $\beta$ -functions and demand that they vanish. In any case it remains technically involved. In general two outcomes are possible:

- if from independent considerations we know that the model at hand is an exact two-dimensional CFT, then we get a solution for all values of  $\alpha'$  (albeit a perturbative one);
- only an  $\alpha'$ -expansion is available.

In this latter case, for the bosonic NS-NS sector we obtain:

$$\beta_{\mu\nu}^G = \alpha' R_{\mu\nu} - \frac{\alpha'}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} + 2\alpha' \nabla_\mu \nabla_\nu \Phi + \mathcal{O}(\alpha'^2 \partial^4), \quad (39a)$$

$$\beta_{\mu\nu}^B = \alpha' \nabla_\lambda H_{\mu\nu}^\lambda - 2\alpha' H_{\mu\nu}^\lambda \nabla_\lambda \Phi + \mathcal{O}(\alpha'^2 \partial^4), \quad (39b)$$

$$\beta^\Phi = \frac{D - D_{\text{cr}}}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\mu \Phi \nabla^\mu \Phi - \frac{\alpha'}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} + \mathcal{O}(\alpha'^2 \partial^4), \quad (39c)$$

where  $D_{\text{cr}}$  is the critical dimension that can be either  $D_{\text{cr}} = 10$  or  $D_{\text{cr}} = 26$ . It is a very remarkable fact that those equations stem from the variation of an effective action:

$$S_{\text{eff}} = \frac{1}{2\kappa_0^2} \int d^{D_{\text{cr}}} x \sqrt{-G} e^{-2\Phi} \left[ R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \mathcal{O}(\alpha'^2 \partial^4) \right]. \quad (40)$$

As we pointed out previously, this action provides both a way to compute the allowed classical backgrounds where the strings consistently propagate and an effective low-energy description of the massless degrees of freedom. Obviously it receives higher-order corrections. Fermions can also be included by computing amplitudes that involve them. In this way the effective action turns out to be a genuine supergravity action. This latter aspect is extremely important because it provides a complete description of how physics at Planck scale (described by string theory) approximates at low-energies as usual field theory.

The above action is written in the so-called *string frame*, *i.e.* with the fields as they appear in the string sigma-model. This is not on the other hand what one would get by a natural generalization of the GR equations. For this reason an equivalent description is usually given in the so-called *Einstein frame* where the fields are defined by

$$\begin{cases} \Omega = \frac{2}{D-2}(\Phi_0 - \Phi) \\ \tilde{G}_{\mu\nu} = e^{2\Omega} G_{\mu\nu} \\ \tilde{\Phi} = \Phi - \Phi_0 \\ \kappa = \kappa_0 e^{\Phi_0} \end{cases} \quad (41)$$

so that the action becomes

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\tilde{G}} \left[ \tilde{R} - \frac{1}{6} \partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} - \frac{1}{12} e^{-\tilde{\Phi}/6} H_{\mu\nu\rho} H^{\mu\nu\rho} + \mathcal{O}(\alpha'^2 \partial^4) \right]. \quad (42)$$

### 5.5. Including Ramond–Ramond fields

The fundamental string, which is the elementary object of the perturbative approach leading to the equations of motion above, does not couple to the Ramond–Ramond fields. In the case of the three-form of type IIB this is because the string is not charged, for all the other forms the very dimension of the string would not allow for any couplings. This means in particular that there is no straightforward way to incorporate the Ramond–Ramond backgrounds in a sigma-model approach of perturbative string theory. This does not mean that those fields are arbitrary. As a matter of fact they still are coherent combinations of elementary string excitations and they must satisfy some equations which can be computed in terms of string amplitudes and then interpreted at low energies as field-theory vertices. In this way, we obtain a low-energy effective description for the RR fields that is given by type I or type II supergravity in ten dimensions.

As an example we write the bosonic sector for the type IIB string, in the Einstein frame:

$$S_{\text{IIB}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left[ R - \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} e^{-\Phi} H_{[3]}^2 - \frac{1}{2} e^{2\Phi} F_{[1]}^2 - \frac{1}{12} e^\Phi \tilde{F}_{[3]}^2 - \frac{1}{48} F_{[5]}^2 \right], \quad (43)$$

where  $\tilde{F}_{[3]} = F_{[3]} - A_{[0]} \wedge H_{[3]}$ . Although we will not expand on this, it is interesting to remark that the couplings of the NS three-form and the RR three-form are S-dual, *i.e.* they get exchanged under  $\Phi \rightarrow -\Phi$ .

Similar expressions exist for type IIA and type I.

## 6. Remarkable solutions

### 6.1. Brane-like solutions

Solving the equations of motion is not in general an easy task. A possible approach consists in making some ansatz and look for special classes. In general, the backgrounds thus obtained will receive higher-order  $\alpha'$  corrections. Some of them will turn out to be exact (this is only possible in presence of pure NS fields) and, at least in the near-horizon limit when the symmetries are enhanced, the corresponding sigma model is an exact CFT.

The most simple ansatz is obtained for vanishing dilaton and antisymmetric field strengths. In this case the equations reduce to  $R_{\mu\nu} = 0$ , *i.e.* the solution must be Ricci flat as it is the case of Calabi–Yau manifolds. Less trivial spacetimes are obtained by taking into account the other fields that, from a general-relativity point of view, will contribute to the energy momentum tensor of the system that will reflect in a non-vanishing curvature. These solutions can then be interpreted as backgrounds created by some classical, electrically or magnetically charged  $Dp$  branes. In particular, if we allow for the dilaton and a single  $F_{[n]}$  form the equations of motion read

$$\begin{cases} R_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{e^{\alpha\Phi}}{2(n-1)!}\left(F_{\mu\mu_2\dots\mu_n}F_{\nu}{}^{\mu_2\dots\mu_n} - \frac{n-1}{8n}F_{\mu_1\dots\mu_n}F^{\mu_1\dots\mu_n}G_{\mu\nu}\right) \\ \nabla_\mu(e^{\alpha\Phi}F_{\mu_2\dots\mu_n}^\mu) = 0 \\ \Delta\Phi = \frac{\alpha}{2n!}e^{\alpha\Phi}F_{\mu_1\dots\mu_n}F^{\mu_1\dots\mu_n}. \end{cases} \quad (44)$$

In the presence of a  $p$  brane, the  $SO(1,9)$  symmetry is broken to  $SO(1,p) \times SO(D-p-1)$ , that is the  $p+1$  longitudinal coordinates are separated from the  $D-p-1$  transverse. The natural ansatz is then

$$ds^2 = e^{2A(r)}\eta_{\alpha\beta}dx^\alpha dx^\beta + e^{2B(r)}\delta_{mn}dy^m dy^n \quad (45a)$$

$$\Phi = \Phi(r), \quad (45b)$$

where  $x^\alpha$  are the longitudinal coordinates,  $y^m$  the transverse and  $r$  is the transverse radius

$$r^2 = \sum_{m=p+1}^{D-1} y^m y^m. \quad (46)$$

The ansatz for  $F_{[n]}$  depends on whether we choose an electric or a magnetic coupling.

*NS5 brane* One of the most remarkable solutions is that obtained in type II (A or B) in presence of an NS5 brane. If we choose a magnetic coupling, corresponding to a solitonic-type brane, from the ansatz above we obtain the solution

$$ds^2 = \frac{1}{H(r)^{1/4}}\left(dx^\alpha dx^\beta \eta_{\alpha\beta} + H(r)dy^m dy^n \delta_{mn}\right) \quad (47a)$$

$$H_{[3]} = -\epsilon_{m_1 m_2 m_3} \partial_r H(r) dy^{m_1} \wedge dy^{m_2} \wedge dy^{m_3} \quad (47b)$$

$$e^\Phi = \sqrt{H(r)}, \quad (47c)$$

where the harmonic function  $H(r)$  is

$$H(r) = 1 + \frac{k}{r^2} \quad (48)$$

with  $k$  the NS5 brane charge. The initial ten-dimensional Poincaré symmetry is clearly broken to  $\text{Poincaré}_6 \times SO(4)$ .

If we consider the geometry close to the brane, *i.e.* in the  $r \rightarrow 0$  limit we get:

$$H(r) \rightarrow \frac{k}{r^2} \quad (49a)$$

$$e^\Phi \rightarrow \frac{\sqrt{k}}{r} \quad (49b)$$

$$ds^2 \rightarrow e^{-\Phi/2} \left( dx^\alpha dx^\beta \eta_{\alpha\beta} + k \frac{dr^2}{r^2} + k d\Omega_3 \right), \quad (49c)$$

where  $kd\Omega_3$  is the line element for a three-sphere of radius  $\sqrt{k}$ <sup>7</sup>. Introducing the variable  $z$  as

$$r = e^{-z/\sqrt{k}}, \quad (50)$$

we observe that the background fields are those of the WZW model on  $SU(2)$  plus a linear dilaton. This is an exact solution of string theory.

Other well-known applications of this kind of backgrounds can be found in the so-called little string theory.

*D5 in type IIB* For a D5 brane in type IIB we can choose a magnetic coupling (getting once more a solitonic brane). In this case one can easily verify that the solution is the same as before with the exchange  $\Phi \rightarrow -\Phi$  and  $H_{[3]} \rightarrow F_{[3]}$ . In other words the two systems are S-dual.

No sigma-model interpretation is known, due to the presence of the Ramond–Ramond field  $F_{[3]}$ .

*D3 in type IIB* Dirichlet branes D3 in type IIB correspond to self-dual  $F_{[5]}$  fluxes. In this case there is no dilaton and the solution keeps a  $\text{Poincaré}_4 \times SO(6)$  symmetry which can be identified with an  $\text{AdS}_5 \times S^5$  geometry (the symmetry is  $SO(2,4) \times SO(6)$ ). Again this is not an exact sigma-model but it has acquired an important role as the simplest framework for implementing the holographic principle. According to the latter, the theory in the bulk (the  $\text{AdS}_5$  background) is equivalent to the  $N = 4$  super-Yang–Mills theory living on its border.

*NS5-F1 system* Another configuration possible in both type II theories is obtained introducing a set of magnetically charged NS5 branes as above and a set of electrically charged fundamental strings. In general this solution will have a dilaton but in the  $r \rightarrow 0$  limit the geometry is simply given by  $\mathbb{R}^4 \times \text{AdS}_3 \times S^3$  with an NS three-form on the curved part and no dilaton. This can be easily identified with the WZW model on the group  $SL(2, \mathbb{R}) \times SU(2)$  and it is simple to show that supersymmetry or – more strongly – Weyl invariance imposes the two curvatures to be equal in modulus.

Again it is possible to make an S-duality transformation and the resulting system can be interpreted as a D5-D1 in type IIB for which no sigma-model description is known.

*M-theory solutions* Ten-dimensional supergravity is not the most general theory available. In fact, although less understood, a more general theory can be formulated, admitting an eleven-dimensional  $N = 1$  SUGRA limit: the M-theory.

In this theory an  $F_{[4]}$  field is present, which couples electrically to an M2 brane or magnetically to a M5 brane. The simplest solution then consists in the M5 brane. This solution has symmetry  $\text{Poincaré}_6 \times SO(5)$  (it is in some sense a generalization of the NS5 case above) and in the near-horizon limit has geometry  $\text{AdS}_7 \times S^4$ , with the four-form flux proportional to the volume form of the sphere.

## 6.2. Constant-curvature spaces

A final remark concerns the appearance of constant-curvature spaces. The solutions above include only spheres or anti-de Sitter spaces. This is not surprising as one can see by considering the following heuristic argument. The Einstein equations in vacuum with a cosmological constant read:

$$R_{\mu\nu} - \frac{R}{2}G_{\mu\nu} + \Lambda G_{\mu\nu} = 0. \quad (51)$$

<sup>7</sup> Notice that for  $r \rightarrow 0$ , the dilaton diverges and we are naturally driven towards a strong-coupling regime.

Taking the trace one obtains

$$R = \Lambda \frac{2D}{D-2}. \quad (52)$$

The solution is therefore a constant-curvature space, which is maximally symmetric:  $\text{AdS}_D$ ,  $\text{dS}_D$ ,  $S^D$  or  $\text{H}_D$ , depending on the sign of  $\Lambda$  and on the signature.

The cosmological constant in the type of system we are considering is effectively simulated by an  $F^2$ -term (see *e.g.* Eq. Equation (43)) which is positive in the Euclidean case and negative in the Minkowskian, leaving as only choices  $S^n$  and  $\text{AdS}_n$ .

The search for de Sitter spaces in supergravity and string theories has a long history. It was noticed long time ago that de Sitter superalgebras lead very often to tachyonic spectra, which are sources of instabilities. Despite that, vacua with positive cosmological constant were found in gauged four-dimensional supergravities. Whether those can be uplifted in some higher-dimensional theory or string theory is however questionable. This issue has recently attracted some attention, but no clear construction for de Sitter-like solutions has emerged in genuine string theory. Similar difficulties appear for hyperbolic spaces, but any further discussion of this subject goes beyond the scope of the present notes.

Let us finally stress that one should not exclude the possibility that searching de Sitter vacua in string theory is of little physical relevance. Firstly because de Sitter per se is not a faithful description of the universe at any time. And, more importantly, because the cosmological constant, as it is observed today, is mainly an infrared effect that integrates all possible scales and phase transitions. As such it is probably not captured by a solution of a first-quantized string theory.

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